Derivation of asymptotic two-dimensional time-dependent equations for surface water wave propagation

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A general method for the derivation of asymptotic nonlinear models in shallow and deep water is presented. Starting from a general dimensionless version of the water wave equations, we reduce the problem to a system of two equations on the surface elevation and the velocity potential at the free surface. These equations involve a Dirichlet–Neumann operator and we show that many asymptotic models can be recovered by a simple analysis of this operator. Based on this method, a new two-dimensional fully dispersive model for small wave steepness is also derived, which extends to an uneven bottom the approach developed by Matsuno [Phys. Rev. E **47**, 4593 (1993)] and Choi [J. Fluid Mech. **295**, 381 (1995)]. This model is still valid in shallow water but with less precision than what can be achieved with the Green–Naghdi model when fully nonlinear waves are considered. The combination, or the coupling, of the new fully dispersive equations with the fully nonlinear shallow water Green–Naghdi equations represents a relevant model for describing ocean wave propagation from deep to shallow waters. © *2009 American Institute of Physics*.
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I. INTRODUCTION

The propagation of surface waves through an incompressible homogeneous inviscid fluid is described by the Euler equations combined with nonlinear boundary conditions at the free surface and at the bottom. This problem is extremely difficult to solve, in particular, because the moving surface boundary is part of the solution. The complexity of this problem led physicists, oceanographers, and mathematicians to derive simpler sets of equations in some specific physical regimes. Equations thus obtained may be divided into two groups, namely, shallow water models and small steepness fully dispersive models. We name the latter "arbitrary depth models" after Matsuno.[1] One of the goals of this paper is to clarify the range of validity of these models and to show why at least two different asymptotic models are necessary for a correct description of ocean water waves.

In shallow water conditions, the classical approach is based on a perturbation method with respect to a small parameter $\mu=(h_0/L)^2$ (h_0 the characteristic water depth, L the characteristic horizontal scale) in order to reduce the three-dimensional equation system to a two-dimensional (2D) one. This method, initially introduced by Boussinesq,[2] allows one to derive several shallow water equations, which are named "Boussinesq-type equations." A large class of such equations can be expressed in the following 2D [throughout this paper, one-dimensional (1D) and 2D refer to the dimensionality of the wave free surface] nondimensional form:

$$\partial_t \zeta + \nabla \cdot (h\overline{\mathbf{v}}) = 0,$$
$$\partial_t \overline{\mathbf{v}} + \varepsilon(\overline{\mathbf{v}} \cdot \nabla)\overline{\mathbf{v}} + \nabla \zeta = \mu \mathcal{D} + O(\mu^2), \quad (1)$$

where $\epsilon=a/h_0$ (a the order of free surface amplitude) is the nonlinearity parameter and the dimensionless flow variables are the water depth h, the surface elevation ζ, and the depth-averaged velocity $\overline{\mathbf{v}}$. $\mathcal{D}$ characterizes nonhydrostatic and dispersive effects and is a function of wave variables and their derivatives. Higher-order Boussinesq equations can be derived (e.g., Ref. 3), but in this paper we restrict our analysis to $O(\mu^2)$, which is a good approximation for most of near-shore wave applications (e.g., Refs. 4–6).

In its classical form, Boussinesq wave theory is a 1D approach based on the assumptions of weak dispersion, weak nonlinearity, and balance between dispersion and nonlinearity: $\epsilon=O(\mu)\ll 1$. The Kortweg–de Vries (KdV) (Ref. 7) and Benjamin–Bona–Mahony (BBM) (Ref. 8) equations can also be derived in the case of unidirectional waves; this approach is relevant to study fundamental wave dynamics problems, such as 1D solitary wave propagation on a flat bottom (e.g., Refs. 2 and 9). However, the classical Boussinesq assumptions may severely restrict applicability to real world wave propagation problems. Applications to coastal zone have motivated theoretical developments for extending the range of applicability of Boussinesq-type equations in terms of varying bottom, dispersive, and nonlinearity effects, which play an important role in the near-shore wave dynamics (see reviews by Dingemans,[10] Madsen and Schäffer,[3] Wu,[11] Kirby,[12] and Barthélemy[13]).

The 2D Boussinesq equations [$\epsilon=O(\mu)\ll 1$] for a nonflat bottom were first derived by Peregrine;[14] in this model, the term $\mathcal{D}$ in the second equation of Eq. (1) is given by

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$$\mathcal{D}=\frac{h}{2}\nabla[\nabla\cdot(h\partial_t\overline{\mathbf{v}})]-\frac{h^2}{6}\nabla^2\partial_t\overline{\mathbf{v}}. \quad (2)$$

For many coastal applications the weak dispersion of these equations is a critical limitation. Witting[15] proposed a method based on Padé expansion of the exact linear phase velocity to improve Boussinesq-type equations. From this method, several equations [order $O(\mu^2)$] with improved dispersion characteristic have been derived (e.g., Refs. 16–20).

In 1953, a breakthrough treating nonlinearity was made by Serre (see Barthélemy[13] for a review). He derived 1D fully nonlinear [$\epsilon=O(1)$] weakly dispersive equations for a horizontal bottom; the nonhydrostatic and dispersive term $\mathcal{D}$ in Eq. (1) is then given by

$$\mathcal{D}=\frac{1}{3h}\partial_x[h^3(\overline{v}_{xt}+\overline{v}\overline{v}_{xx}-(\overline{v}_x)^2)]. \quad (3)$$

The same system was obtained later by Su and Gardner.[21] Seabra-Santos *et al.*[22] provided an extension of this model to a nonflat bottom:

$$\mathcal{D}=-\frac{1}{h}\left\{\partial_x\left[h^2\left(\frac{1}{3}P+\frac{1}{2}Q\right)\right]+b_xh\left(\frac{1}{2}P+Q\right)\right\}, \quad (4)$$

with $P=-h[\overline{v}_{xt}+\overline{v}\overline{v}_{xx}-(\overline{v}_x)^2]$ and $Q=b_x(\overline{v}_t+\overline{v}\overline{v}_x+b_{xx}\overline{v}^2)$. Dingemans[10] (p. 613) expressed mistrust for the validity of these equations for *uneven topographies* considering that the derivation required the assumption of vertical uniformity of the horizontal velocity. In fact, the asymptotic derivation of Su and Gardner[21] for a flat bottom shows how such vertical uniformity assumption is not required, and Cienfuegos *et al.*[23] showed how to extend this to nonflat bottoms.

Finally, Green and Naghdi[24] derived 2D fully nonlinear weakly dispersive equations for an uneven bottom which represent a 2D extension of the Serre equations. Except for being formulated in terms of the velocity vector at an arbitrary z level, the equations of Wei *et al.*[25] are basically equivalent to the 2D Serre or Green–Naghdi equations; the equations derived in Ref. 26 through Hamilton's principle are also exactly the same as the ones derived in Ref. 24 and thus the same as Eq. (26) derived below, though this might not seem obvious at first sight; see also Ref. 27 for time-dependent bottom topographies.

The range of validity of all the models introduced above may vary as far as the nonlinearity parameter ε is concerned, but they all require that the shallowness parameter μ be small. In deeper water ($\mu\not\ll 1$) the Boussinesq wave theory fails but it is yet possible to derive asymptotic expansions from the water wave equations under the condition that the *steepness* $\varepsilon\sqrt{\mu}=a/L$ is small. Such an approach was made in 1D and flat bottoms by Matsuno[28] and extended to uneven bottoms,[29] 2D weakly transverse waves,[1] and higher-order expansions.[30] Since one always has $\varepsilon\lesssim 1$, the small steepness assumption $\varepsilon\sqrt{\mu}\ll 1$ is also satisfied in the shallow water regime $\mu\ll 1$ discussed above and this is the reason why it is often claimed that the models derived in Refs. 1 and 28–30 are valid in the whole range $\mu\in(0,\infty)$. However, as we show here, their precision is then far below the one of the Green–Naghdi equations.

In this paper we propose a systematic derivation of all the models evoked in this introduction (and of some new ones); to this end, we use the global method introduced in the recent mathematical work.[31] Starting from a general nondimensionalized version of the water wave equations (which takes into account the different nondimensionalizations used in deep and shallow water), following Refs. 32 and 33 we reduce the problem to a system of two equations on the surface elevation and the velocity potential at the free surface. These equations involve a Dirichlet–Neumann operator and we show that all the asymptotic models can be recovered by a simple analysis of this operator. In the arbitrary depth setting (i.e., if we do not assume that $\mu\ll 1$) we give an asymptotic expansion of this operator with respect to ε; in the shallow water setting, we rather choose to use an exact expression of the Dirichlet–Neumann operator in terms of the averaged velocity $\overline{\mathbf{v}}$ and to provide an asymptotic expansion of the velocity potential in terms of μ since ε is not always small (e.g., $\varepsilon\sim 1$ for Green–Naghdi).

As said above, all the asymptotic models are obtained in a systematic way with the present approach. In addition, we derive a new fully dispersive model for 2D surface waves over an uneven bottom and arbitrary depth. Our method brings clarifications about the validity domain of many asymptotic wave models, in particular, the following.

- We can easily show that the shallow water limit of the Dirichlet–Neumann expansion corresponding to the arbitrary depth models has a very poor precision in the fully nonlinear case $\varepsilon\sim 1$; this proves that the arbitrary depth models such as those of Refs. 1 and 28–30 are not in general suitable for nonlinear shallow water waves.
- We do not make any assumption (other than irrotationality) on the velocity profile or on any related quantity. The only assumptions we make concern the value of the parameters ε and μ (and a third parameter β linked to the amplitude of the bottom variations for uneven bottoms); the properties of the velocity profile (and, in particular, its vertical behavior) are then rigorously established; we thus believe that this article should clarify the discussions concerning the assumptions made on the velocity profile, and, in particular, the controversy raised in Ref. 10 about the Serre equations.

II. GENERAL NONDIMENSIONALIZED WATER WAVE EQUATIONS

Parametrizing the free surface by $z=\zeta(t,X)$ [with $X=(x,y)\in\mathbb{R}^2$] and the bottom by $z=-h_0+b(X)$ (with $h_0>0$ constant), one can use the incompressibility and irrotationality conditions to write the water wave equations under Bernouilli's formulation in terms of a potential velocity ϕ (i.e., the velocity field is given by $\mathbf{V}=\nabla_{X,z}\phi$):

$$
\begin{aligned}
&\partial_x^2\phi+\partial_y^2\phi+\partial_z^2\phi=0, \quad -h_0+b\le z\le\zeta,\\
&\partial_n\phi=0, \quad z=-h_0+b,\\
&\partial_t\zeta+\nabla\zeta\cdot\nabla\phi=\partial_z\phi, \quad z=\zeta,\\
&\partial_t\phi+\tfrac{1}{2}[|\nabla\phi|^2+(\partial_z\phi)^2]+g\zeta=0, \quad z=\zeta,
\end{aligned}
\tag{5}
$$

where g is the gravitational acceleration, $\nabla=(\partial_x,\partial_y)^T$, and $\partial_n\phi$ is the outward normal derivative at the boundary of the fluid domain. Surface tension has no appreciable influence for the phenomena under consideration here and has therefore been neglected; our approach can, however, be easily generalized to the case of nonzero surface tension.

In deriving approximate equations by asymptotic methods it is necessary to introduce dimensionless quantities based on characteristic scales for the wave motion. Four main length scales are involved in this problem: the characteristic water depth h_0, the characteristic horizontal scale L, the order of free surface amplitude a, and the order of bottom topography variation B. Three independent dimensionless parameters can be formed from these four scales. We choose

$$
\frac{a}{h_0}=\epsilon, \quad \frac{h_0^2}{L^2}=\mu, \quad \frac{B}{h_0}=\beta, \tag{6}
$$

where ϵ is often called the nonlinearity parameter, while μ is the shallowness parameter.

Commonly two distinct nondimensionalizations are used in oceanography (e.g., Ref. 10) depending on the value of μ: shallow water scaling ($\mu\ll 1$) and Stokes wave scaling for intermediate to deep water ($\mu\not\ll 1$). In this paper, we present as in Refs. 1, 28, and 29 a general nondimensionalization which applies to any wave regime. The order of magnitude of wave motion variables are given by the linear wave theory. In particular, $(gh_0\nu)^{1/2}$ and $(aL/h_0)(gh_0/\nu)^{1/2}$ are the characteristic scales of, respectively, the wave celerity and the potential velocity, with ν a dimensionless parameter which can be expressed as

$$
\nu=\tanh(\mu^{1/2})/\mu^{1/2}.
$$

Let us normalize all variables according to the scales anticipated on physical grounds:

$$
x=Lx', \quad y=Ly', \quad z=h_0\nu z', \quad t=\frac{L}{\sqrt{gh_0\nu}}t',
$$

$$
\zeta=a\zeta', \quad \Phi=\frac{a}{h_0}L\sqrt{\frac{gh_0}{\nu}}\Phi', \quad b=Bb'.
$$

We can recover the classical scalings for shallow and deep water from this general nondimensionalization: $\nu\sim 1$ when $\mu\ll 1$ and $\nu\sim\mu^{-1/2}$ when $\mu\gg 1$.

The equations of motion (5) then become (after dropping the primes for the sake of clarity)

$$
\begin{aligned}
&\nu^2\mu\partial_x^2\Phi+\nu^2\mu\partial_y^2\Phi+\partial_z^2\Phi=0, \quad \frac{1}{\nu}(-1+\beta b)\le z\le\frac{\varepsilon}{\nu}\zeta,\\
&-\nu^2\mu\nabla\left(\frac{\beta}{\nu}b\right)\cdot\nabla\Phi+\partial_z\Phi=0, \quad z=\frac{1}{\nu}(-1+\beta b),\\
&\partial_t\zeta-\frac{1}{\mu\nu^2}\left[-\nu^2\mu\nabla\left(\frac{\varepsilon}{\nu}\zeta\right)\cdot\nabla\Phi+\partial_z\Phi\right]=0, \quad z=\frac{\varepsilon}{\nu}\zeta,\\
&\partial_t\Phi+\frac{1}{2}\left[\frac{\varepsilon}{\nu}|\nabla\Phi|^2+\frac{\varepsilon}{\mu\nu^3}(\partial_z\Phi)^2\right]+\zeta=0, \quad z=\frac{\varepsilon}{\nu}\zeta.
\end{aligned}
\tag{7}
$$

In order to reduce this set of equations into a system of two evolution equations, we introduce the trace of the velocity potential at the free surface, namely, $\psi=\Phi_{|_{z=(\varepsilon/\nu)\zeta}}$ and the Dirichlet–Neumann operator $\mathcal{G}_\mu^\nu[(\varepsilon/\nu)\zeta,\beta b]\cdot$ as

$$
\mathcal{G}_\mu^\nu\left[\frac{\varepsilon}{\nu}\zeta,\beta b\right]\psi=-\nu^2\mu\nabla\left(\frac{\varepsilon}{\nu}\zeta\right)\cdot\nabla\Phi_{|_{z=(\varepsilon/\nu)\zeta}}+\partial_z\Phi_{|_{z=(\varepsilon/\nu)\zeta}}, \tag{8}
$$

with Φ solving the boundary value problem

$$
\nu^2\mu\partial_x^2\Phi+\nu^2\mu\partial_y^2\Phi+\partial_z^2\Phi=0, \quad \frac{1}{\nu}(-1+\beta b)\le z\le\frac{\varepsilon}{\nu}\zeta,
$$

$$
\Phi_{|_{z=\varepsilon/\nu\zeta}}=\psi, \quad \partial_n\Phi_{|_{z=1/\nu(-1+\beta b)}}=0.
$$

One can check that $\mathcal{G}_\mu^\nu[(\varepsilon/\nu)\zeta,\beta b]\psi=\sqrt{1+|\nabla[(\varepsilon/\nu)\zeta]|^2}\partial_n\Phi_{|_{z=(\varepsilon/\nu)\zeta}}$, where $\partial_n\Phi$ stands for the upward nondimensionalized normal derivative at the surface.

As remarked in Refs. 32 and 33 Eq. (7) is equivalent to a set of two equations on the free surface parametrization ζ and the trace of the velocity potential at the surface $\psi=\Phi_{|_{z=(\varepsilon/\nu)\zeta}}$ involving the Dirichlet–Neumann operator. Namely,

$$
\begin{aligned}
&\partial_t\zeta-\frac{1}{\mu\nu^2}\mathcal{G}_\mu^\nu\left[\frac{\varepsilon}{\nu}\zeta,\beta b\right]\psi=0,\\
&\partial_t\psi+\zeta+\frac{\varepsilon}{2\nu}|\nabla\psi|^2\\
&\qquad-\frac{\varepsilon\mu}{\nu^3}\frac{\left[\frac{1}{\mu}\mathcal{G}_\mu^\nu\left[\frac{\varepsilon}{\nu}\zeta,\beta b\right]\psi+\nu\nabla(\varepsilon\zeta)\cdot\nabla\psi\right]^2}{2(1+\varepsilon^2\mu|\nabla\zeta|^2)}=0.
\end{aligned}
\tag{9}
$$

In order to simplify this system and to make the dependence on the parameter ν more transparent, let us define the operator $\mathcal{G}_\mu[\varepsilon\zeta,\beta b]$ obtained by taking formally $\nu=1$ in Eq. (8), namely,

$$
\mathcal{G}_\mu[\varepsilon\zeta,\beta b]\psi=-\mu\nabla(\varepsilon\zeta)\cdot\nabla\Phi_{|_{z=\varepsilon\zeta}}+\partial_z\Phi_{|_{z=\varepsilon\zeta}}, \tag{10}
$$

with Φ solving the boundary value problem

$$\mu\partial_x^2\Phi + \mu\partial_y^2\Phi + \partial_z^2\Phi = 0, \quad -1 + \beta b \le z \le \varepsilon\zeta,$$
$$\Phi_{|_{z=\varepsilon\zeta}} = \psi, \quad \partial_n\Phi_{|_{z=-1+\beta b}} = 0. \tag{11}$$

Easy computations show that the operators $\mathcal{G}_\mu^\nu[(\varepsilon/\nu)\zeta,\beta b]$ and $\mathcal{G}_\mu[\varepsilon\zeta,\beta b]$ defined in Eqs. (8) and (10) are linked through the identity

$$\mathcal{G}_\mu[\varepsilon\zeta,\beta b] = \frac{1}{\nu}\mathcal{G}_\mu^\nu\left[\frac{\varepsilon}{\nu}\zeta,\beta b\right]. \tag{12}$$

Using Eq. (12), one can therefore transform Eq. (9) into

$$\partial_t\zeta - \frac{1}{\mu\nu}\mathcal{G}_\mu[\varepsilon\zeta,\beta b]\psi = 0,$$
$$\partial_t\psi + \zeta + \frac{\varepsilon}{2\nu}|\nabla\psi|^2 - \frac{\varepsilon\mu}{\nu}\frac{\left[\frac{1}{\mu}\mathcal{G}_\mu[\varepsilon\zeta,\beta b]\psi + \nabla(\varepsilon\zeta)\cdot\nabla\psi\right]^2}{2(1+\varepsilon^2\mu|\nabla\zeta|^2)} = 0. \tag{13}$$

System (13) recasts the water wave equations in terms of the free surface elevation ζ and the velocity potential ψ at the surface; this is the formulation which will serve as a basis for all the computations in this article—of course, the first equation of Eq. (13) can be written under the more usual form $\partial_t\zeta + \nabla\cdot(h\bar{\mathbf{v}})$, where $\bar{\mathbf{v}}$ is the vertically averaged horizontal velocity, see Eq. (16) below.

III. SHALLOW WATER MODELS

This section is devoted to the study of shallow water waves: $\mu \ll 1$. From the assumption $\mu \ll 1$, one gets $\nu \sim 1$ and we take $\nu = 1$ in Eq. (13) to simplify (this corresponds to the usual shallow water nondimensionalization).

The most general situation (when no assumption is made on ε and β) is addressed in Sec. III A, where the Green–Naghdi equations (also called Serre[34] or *fully* nonlinear Boussinesq equations[25]) are derived. The key point is that $\mathcal{G}_\mu[\varepsilon\zeta,\beta b]$ and $\nabla\psi$ are respectively related to the averaged velocity $\bar{\mathbf{v}}$ by an exact formula and an expansion with respect to μ.

We then briefly show how to recover simpler models under additional assumptions on ε; the *moderately* nonlinear case $\varepsilon = O(\sqrt{\mu})$ is addressed in Sec. III B and the *weakly* nonlinear case $\varepsilon = O(\mu)$ in Sec. III C. Some new models are then derived in the next sections.

A. The fully nonlinear case: $\epsilon \sim 1$, $\beta \sim 1$, and $\mu \ll 1$

1. The Dirichlet–Neumann operator and the averaged velocity

Let us write the depth-averaged horizontal velocity $\bar{\mathbf{v}}$ defined in terms of the solution Φ to the Laplace equation (11),

$$\bar{\mathbf{v}}(t,X) = \frac{1}{h(t,X)}\int_{-1+\beta b(X)}^{\varepsilon\zeta(t,X)} \nabla\Phi(t,X,z)dz, \tag{14}$$

[recall that $X=(x,y)$, $\nabla = (\partial_x,\partial_y)^T$, and $h = 1+\varepsilon\zeta-\beta b$]. Using Green's identity and definition (10) of the operator $\mathcal{G}_\mu[\varepsilon\zeta,\beta b]$ one deduces from Eq. (14) that

$$\mathcal{G}_\mu[\varepsilon\zeta,\beta b]\psi = -\mu\nabla\cdot(h\bar{\mathbf{v}}). \tag{15}$$

Replacing $\mathcal{G}_\mu[\varepsilon\zeta,\beta b]\psi$ by this expression in the first equation of Eq. (13), one recovers the classical formulation of the kinematic condition in terms of $\bar{\mathbf{v}}$,

$$\partial_t\zeta + \nabla\cdot(h\bar{\mathbf{v}}) = 0. \tag{16}$$

2. Asymptotic expansion of $\nabla\psi$

Since $\mu \ll 1$, we look for an asymptotic expansion of Φ under the form

$$\Phi_{\mathrm{app}} = \sum_{j=0}^{N}\mu^j\Phi_j. \tag{17}$$

Plugging this expression into the Laplace equation (11) one can cancel the residual up to the order $O(\mu^{N+1})$ provided that

$$\forall j = 0,\ldots,N, \quad \partial_z^2\Phi_j = -\partial_x^2\Phi_{j-1} - \partial_y^2\Phi_{j-1} \tag{18}$$

(with the convention that $\Phi_{-1}=0$), together with the boundary conditions

$$\forall j = 0,\ldots,N, \quad \begin{cases}\Phi_{j|_{z=\varepsilon\zeta}} = \delta_{0,j}\psi \\ \partial_z\Phi_j = \beta\nabla b\cdot\nabla\Phi_{j-1|_{z=-1+\beta b}}\end{cases}, \tag{19}$$

(where $\delta_{0,j}=1$ if $j=0$ and 0 otherwise).

Solving the ordinary differential equation (18) with Eq. (19) is completely straightforward, and this procedure can be implemented on any symbolic computation software to compute the Φ_j at any order. For our purposes here, we must take $N=1$ and thus need to compute Φ_0 and Φ_1; one finds

$$\Phi_0 = \psi, \tag{20}$$

$$\Phi_1 = (z-\varepsilon\zeta)\left[-\tfrac{1}{2}(z+\varepsilon\zeta) - 1 + \beta b\right]\Delta\psi + \beta(z-\varepsilon\zeta)\nabla b\cdot\nabla\psi. \tag{21}$$

Remark 3.1: *This shows that the velocity potential (and thus the velocity field) does not depend on z at leading order; if we include the* $O(\mu^2)$ *term and without assumption on* ε, *the potential depends quadratically on z, as is well known.*

Remark 3.2: *In the case of a flat bottom* $(b=0)$ *and a flat surface* $(\zeta=0)$, *the Laplace equation (11) can be solved explicitly; the Fourier transform with respect to the horizontal variables of* Φ *is then given by*

$$\hat{\Phi}(\xi,z) = \frac{\cosh[\sqrt{\mu}(z+1)|\xi|]}{\cosh(\sqrt{\mu}|\xi|)}\hat{\psi}(\xi)$$

and the Fourier transform of Eq. (17) is therefore equal to the Nth order Taylor expansion of this formula. Quite obviously, the range of validity of Eq. (17) is thus restricted to relatively small values of $\sqrt{\mu}|\xi|$ *and therefore to small wavenumbers* ξ *when* μ *is large; this is the reason why we use another approximation in Sec. IV where arbitrary depth is considered.*

An approximation of order $O(\mu^{N+1})$ of the averaged velocity $\overline{\mathbf{v}}$ defined in Eq. (14) is then given by

$$\overline{\mathbf{v}} = \frac{1}{h(t,X)}\int_{-1+\beta b(X)}^{\varepsilon\zeta(t,X)} \nabla\Phi_{\mathrm{app}}(t,X,z)dz + O(\mu^{N+1}) \tag{22}$$

for $N=1$. We thus obtain from Eqs. (17), (20), and (21)

$$\overline{\mathbf{v}} = \nabla\psi - \frac{\mu}{h}\mathcal{T}[h,\beta b]\nabla\psi + O(\mu^2), \tag{23}$$

where $h=1+\varepsilon\zeta-\beta b$ and where the linear operator $\mathcal{T}[h,\beta b]$ is defined as

$$\mathcal{T}[h,\beta b]W = -\tfrac{1}{3}\nabla(h^3\nabla\cdot W) + \beta\tfrac{1}{2}[\nabla(h^2\nabla b\cdot W) - h^2\nabla b\nabla\cdot W] + \beta^2 h\nabla b\nabla b\cdot W. \tag{24}$$

Inverting Eq. (23), we can write $\nabla\psi$ in terms of $\overline{\mathbf{v}}$,

$$\nabla\psi = \overline{\mathbf{v}} + \frac{\mu}{h}\mathcal{T}[h,\beta b]\overline{\mathbf{v}} + O(\mu^2). \tag{25}$$

3. Derivation of the Green–Naghdi equations

We can now derive the Green–Naghdi equations (also called the Serre equations) from the full water wave equations (13), proceeding as follows.

(1) Replace the first equation of Eq. (13) by Eq. (16).
(2) Take the gradient of the second equation of Eq. (13), use Eqs. (15) and (25) to express $\mathcal{G}_\mu[\varepsilon\zeta,\beta b]\psi$ and $\nabla\psi$ in terms of $\overline{\mathbf{v}}$, and neglect all the $O(\mu^2)$ terms.

One thus obtains the Green–Naghdi equations

$$\begin{cases} \partial_t\zeta + \nabla\cdot(h\overline{\mathbf{v}}) = 0, \\ \left(1+\frac{\mu}{h}\mathcal{T}[h,\beta b]\right)\partial_t\overline{\mathbf{v}} + \nabla\zeta + \varepsilon(\overline{\mathbf{v}}\cdot\nabla)\overline{\mathbf{v}} + \mu\varepsilon\left\{-\frac{1}{3h}\nabla\{h^3[(\overline{\mathbf{v}}\cdot\nabla)(\nabla\cdot\overline{\mathbf{v}}) - (\nabla\cdot\overline{\mathbf{v}})^2]\} + \mathcal{Q}[h,\beta b](\overline{\mathbf{v}})\right\} = 0, \end{cases} \tag{26}$$

where the purely topographical term $\mathcal{Q}[h,\beta b](\overline{\mathbf{v}})$ (which is quadratic in $\overline{\mathbf{v}}$) is defined as

$$\mathcal{Q}[h,\beta b](\overline{\mathbf{v}}) = \frac{\beta}{2h}\{\nabla[h^2(\overline{\mathbf{v}}\cdot\nabla)^2 b] - h^2[(\overline{\mathbf{v}}\cdot\nabla)(\nabla\cdot\overline{\mathbf{v}}) - (\nabla\cdot\overline{\mathbf{v}})^2]\nabla b\} + \beta^2[(\overline{\mathbf{v}}\cdot\nabla)^2 b]\nabla b. \tag{27}$$

Remark 3.3: *Formulation (26) of the Green–Naghdi equation is not at first sight the same as usual (see Refs. 24 and 26). Tedious but simple computations show, however, that they are exactly the same, as expected. It can, in particular, be checked quite easily that Eqs. (1.2)–(1.4) of Ref. 26 coincide with Eqs. (26) and (27) when the bottom is flat* ($b=0$).

The interest of the present form is that the operator $[1+(\mu/h)\mathcal{T}[h,\beta b]]$ *in front of* $\partial_t\overline{\mathbf{v}}$ *is that it induces regularizing effects and is thus expected to ease numerical computations (work in progress).*

Remark 3.4: *In Ref. 25, Wei et al. derived some Green–Naghdi equations (fully nonlinear Boussinesq models in that reference) with improved frequency dispersion by replacing the vertically averaged horizontal velocity* $\overline{\mathbf{v}}$ *by the velocity* $\mathbf{v}_\alpha$ *taken at some intermediate depth* $z_\alpha(x,y)$ *(thus following the approach of Nwogu*[17] *for weakly nonlinear Boussinesq systems). Such systems could of course be derived similarly from Eq. (26).*

Remark 3.5: *It is also possible to give a Lie–Poisson Hamiltonian form to the Green–Naghdi equations [see Ref. 35 and Eq. (4.17) of Ref. 36]. Note also that if one neglects the* $O(\mu)$ *terms then Eq. (26) reduces to the standard nonlinear shallow water equations.*

B. The moderately nonlinear case: $\mu \ll 1$ and $\varepsilon = O(\sqrt{\mu})$

The simplifications that can be made on the Green–Naghdi equations (26) under the assumption $\varepsilon=O(\sqrt{\mu})$ depend on the topography. As for the surface variations, we distinguish three different regimes:

(a) Fully nonlinear topography: $\beta=O(1)$. In this case, no significant simplification can be made, and the full equations must be kept.
(b) Moderately nonlinear topography: $\beta=O(\sqrt{\mu})$. In this regime, the third line of Eq. (26) can be written as

$$-\frac{\mu\varepsilon}{3}\nabla\{[(\overline{\mathbf{v}}\cdot\nabla)(\nabla\cdot\overline{\mathbf{v}}) - (\nabla\cdot\overline{\mathbf{v}})^2]\} + O(\mu^2),$$

so that one can replace Eq. (26) by

$$\partial_t\zeta + \nabla\cdot(h\overline{\mathbf{v}}) = 0,$$

$$\left(1+\frac{\mu}{h}\mathcal{T}[h,\beta b]\right)\partial_t\overline{\mathbf{v}} + \nabla\zeta + \varepsilon(\overline{\mathbf{v}}\cdot\nabla)\overline{\mathbf{v}} - \frac{\mu\varepsilon}{3}\nabla\{[(\overline{\mathbf{v}}\cdot\nabla)(\nabla\cdot\overline{\mathbf{v}}) - (\nabla\cdot\overline{\mathbf{v}})^2]\} = 0.$$

Note that some simplifications could also be made in the term $(\mu/h)\mathcal{T}[h,\beta b]\partial_t\overline{\mathbf{v}}$ but they are not interesting because they would partially destroy the regularizing effects evoked in Remark 3.3.

(c) Weakly nonlinear topography: $\beta=O(\mu)$. This stronger assumption allows a simplification of the term $(\mu/h)\mathcal{T}[h,\beta b]\partial_t\overline{\mathbf{v}}$, and one gets

$$\begin{cases} \partial_t\zeta + \nabla\cdot(h\overline{\mathbf{v}}) = 0, \\ \left[1-\frac{\mu}{3h}\nabla(h^3\nabla\cdot)\right]\partial_t\overline{\mathbf{v}} + \nabla\zeta + \varepsilon(\overline{\mathbf{v}}\cdot\nabla)\overline{\mathbf{v}} - \frac{\mu\varepsilon}{3}\nabla\{[(\overline{\mathbf{v}}\cdot\nabla)(\nabla\cdot\overline{\mathbf{v}}) - (\nabla\cdot\overline{\mathbf{v}})^2]\} = 0. \end{cases} \tag{28}$$

The main interest of this model is that in the case of flat bottoms ($b=0$) and surface dimension equal to 1 (no dependence on y), its unidirectional limit gives the Camassa–Holm equation.[37–39] More precisely, it has been rigorously proved in Ref. 40 that one can build an approximate solution of order $O(\mu^2)$ to Eq. (28) under the form

$$\bar{v} = u + \frac{\mu}{12}u_x x + \frac{\mu\varepsilon}{6}uu_{xx},$$

$$\zeta = \bar{v} + \frac{\varepsilon}{4}\bar{v}^2 + \frac{\mu}{6}\bar{v}_{xt} - \varepsilon\mu\left[\frac{1}{6}\bar{v}\bar{v}_{xx} + \frac{5}{48}\bar{v}_x^2\right],$$

where u solves the Camassa–Holm equation

$$u_t + u_x + \tfrac{3}{2}\varepsilon uu_x - \mu\left(\tfrac{1}{4}u_{xxx} + \tfrac{5}{12}u_{xxt}\right) = \tfrac{5}{24}\varepsilon\mu(uu_{xxx} + 2u_x u_{xx}).$$

C. The weakly nonlinear case: $\mu \ll 1$ and $\varepsilon = O(\mu)$

The assumption $\varepsilon = O(\mu)$ is the classical long wave assumption which yields the usual Boussinesq models. Here again, we briefly show how to recover these models in different topographic regimes.

(a) Fully nonlinear topography: $\beta = O(1)$. Neglecting in Eq. (26) the terms which are of order $O(\mu^2)$ under the assumption $\varepsilon = O(\mu)$ yields the equations

$$\partial_t\zeta + \nabla \cdot (h\bar{\mathbf{v}}) = 0,$$

$$\left(1 + \frac{\mu}{h_b}\mathcal{T}[h_b, \beta b]\right)\partial_t\bar{\mathbf{v}} + \nabla\zeta + \varepsilon(\bar{\mathbf{v}} \cdot \nabla)\bar{\mathbf{v}} = 0,$$

with $h_b = 1 - \beta b$. These equations are equivalent to Eqs. (1) and (2) and correspond to the Boussinesq model for nonflat bottoms derived by Peregrine.[14] For many coastal applications the weak dispersion of these equations is a critical limitation. Several alternative equations, with the same approximations $\mu \ll 1$ and $\varepsilon = O(\mu)$, have been developed to improve dispersion properties (e.g., Refs. 16–18 or Refs. 19 and 20) and deal with weakly transverse waves.[41]

(b) Weakly nonlinear topography: $\beta = O(\mu)$. In this case, the equations simplify further into

$$\partial_t\zeta + \nabla \cdot (h\bar{\mathbf{v}}) = 0,$$
$$\left(1 - \frac{\mu}{3}\Delta\right)\partial_t\bar{\mathbf{v}} + \nabla\zeta + \varepsilon(\bar{\mathbf{v}} \cdot \nabla)\bar{\mathbf{v}} = 0 \qquad (29)$$

for which topographic effects play a role only through the presence of $h = 1 + \varepsilon\zeta - \beta b$ in the first equation. As for Eq. (28), it is of interest to study the unidirectional limit of Eq. (29) in the case of flat bottoms ($b = 0$) and surface dimension equal to 1. Neglecting the $O(\varepsilon\mu)$ terms [since $O(\varepsilon\mu) = O(\mu^2)$ in the present scaling] of the Camassa–Holm approximation described in Sec. III B regime (c), an approximate solution of order $O(\mu^2)$ to Eq. (29) is then given by

$$\bar{\mathbf{v}} = u + \frac{\mu}{12}u_x x, \quad \zeta = \bar{\mathbf{v}} + \frac{\varepsilon}{4}\bar{\mathbf{v}}^2 + \frac{\mu}{6}\bar{\mathbf{v}}_{xt},$$

where u solves the BBM-type equation

$$u_t + u_x + \tfrac{3}{2}\varepsilon uu_x - \mu\left(\tfrac{1}{4}u_{xxx} + \tfrac{5}{12}u_{xxt}\right) = 0.$$

As is well known (e.g., Ref. 10), writing $u(t,x) = \underline{u}(\varepsilon t, x - t)$ and denoting $\tau = \varepsilon t$, $\xi = x - t$, and $S = \varepsilon/\mu$, one obtains the KdV equation

$$\underline{u}_\tau + \frac{3}{2}\underline{u}\underline{u}_x + \frac{1}{6S}\underline{u}_{xxx} = 0.$$

When the Stokes number is fixed, this equation is *independent* of ε and μ, which is of course of high interest for numerical simulations since computations must be performed on a much shorter time scale.

IV. FULLY DISPERSIVE MODELS: $\varepsilon\sqrt{\mu} \ll 1$ AND $\epsilon \ll 1$

Small steepness asymptotics ($\varepsilon\sqrt{\mu} \ll 1$) have been used by Matsuno[28] and generalized in Refs. 1, 29, and 30. It is often claimed that the derived equations are valid under the condition $\varepsilon\sqrt{\mu} \ll 1$ only, and that the other asymptotic models (which satisfy this condition) can be deduced from them. We show here that this is not always the case. The unified framework used in this article allows us to check, for instance, that the Matsuno equations do not degenerate into the Green–Naghdi equations in shallow water and that their precision is much smaller.

After making a new asymptotic expansion of the Dirichlet–Neumann operator in this physical regime, we also derive a new generalization of the Matsuno equations for 2D surface waves with an uneven bottom.

A. Asymptotic expansion of the Dirichlet–Neumann operator

When ε and β are small, it is possible to make a Taylor expansion of $\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi$ with respect to ε and β; such an expansion has been derived for 1D surfaces in Refs. 42; this method could be generalized to 2D surfaces, but we chose here to use another technique based on the following formulas:

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi - \mathcal{G}_\mu[0, \beta b]\psi) = -\mathcal{G}_\mu[0, \beta b] \times (\zeta\mathcal{G}_\mu[0, \beta b]\psi) - \mu\nabla\cdot[\zeta(\nabla\psi)]$$

and

$$\lim_{\beta\to 0}\frac{1}{\beta}(\mathcal{G}_\mu[0, \beta b]\psi - \mathcal{G}_\mu[0,0]\psi) = \mu\,\mathrm{sech}(\sqrt{\mu}|D|) \times [\nabla\cdot\{b[\mathrm{sech}(\sqrt{\mu}|D|)\nabla\psi]\}], \qquad (30)$$

where we used the Fourier multiplier notation: given two functions f and u and denoting by $\hat{}$ the Fourier transform, $f(D)u$ is defined as

$$\forall\xi \in \mathbb{R}^2, \quad \widehat{f(D)u}(\xi) := f(\xi)\hat{u}(\xi).$$

We only prove the second of these formulas because the first one can be established with the same techniques and can also be found in the literature (e.g., Ref. 43 and also

Theorem 3.20 of Ref. 44 and Theorem 3.1 of Ref. 31 where a similar formula is established for non-necessarily flat surfaces).

First recall that by definition, $\mathcal{G}_\mu[0,\beta b]\psi=\partial_z\Phi[\beta b]_{|_{z=0}}$, where $\Phi[\beta b]$ solves

$$\mu\Delta\Phi[\beta b]+\partial_z^2\Phi[\beta b]=0 \quad \text{in} -1+\beta b<z<0,$$
$$\Phi[\beta b]_{|_{z=0}}=\psi, \quad (\partial_z\Phi[\beta b]-\mu\beta\nabla\Phi[\beta b]\cdot\nabla b)_{|_{z=-1+\beta b}}=0. \tag{31}$$

We denote by $\Phi[\beta b]$ instead of Φ to emphasize the dependence on the bottom topography. We thus have, for all smooth function φ compactly supported in $\bar{\Omega}$ (with $\bar{\Omega}=\{(X,z),-1+\beta b(X)\leq z\leq 0\}$)

$$\int_\Omega(\mu\Delta\Phi[\beta b]+\partial_z^2\Phi[\beta b])\varphi dXdz + \int_{z=0}(\Phi[\beta b]-\psi)\varphi dX+\int_{z=-1+\beta b}(\partial_z\Phi[\beta b] - \beta\mu\nabla b\cdot\Phi[\beta b])\varphi dX=0.$$

Denoting $\chi=\lim_{\beta\to 0}(1/\beta)(\Phi[\beta b]-\Phi[0])$, one can differentiate the above identity to obtain

$$\int_\Omega(\mu\Delta\chi+\partial_z^2\chi)\varphi dXdz + \int_{z=0}\chi\varphi dX+\int_{z=-1}(\partial_z\chi-\mu\nabla b\cdot\nabla\Phi[0] + b\partial_z^2\Phi[0])\varphi dX=0.$$

Since this identity holds for all test function φ and since $\partial_z^2\Phi[0]=-\mu\Delta\Phi[0]$, we deduce that χ solves the boundary value problem

$$\mu\Delta\chi+\partial_z^2\chi=0 \quad \text{in} -1<z<0,$$
$$\chi_{|_{z=0}}=0, \quad \partial_z\chi_{|_{z=-1}}=\mu\nabla\cdot(b\nabla\Phi[0]_{|_{z=-1}}).$$

This problem can be solved explicitly:

$$\chi(\cdot,z)=\sqrt{\mu}\frac{\sinh(\sqrt{\mu}z|D|)}{\cosh(\sqrt{\mu}|D|)}\frac{\nabla}{|D|}\cdot(b\nabla\Phi[0]_{|_{z=-1}}).$$

Since moreover one has $\lim_{\beta\to 0}(1/\beta)(\mathcal{G}_\mu[0,\beta b]\psi-\mathcal{G}_\mu[0,0]\psi)=\partial_z\chi_{|_{z=0}}$ and that $\Phi[0](\cdot,z)=\{\cosh[\sqrt{\mu}(z+1)|D|]\}/[\cosh(\sqrt{\mu}|D|)]\psi$, we get formula (30).

A first order Taylor expansion of $\mathcal{G}_\mu[\varepsilon\zeta,\beta b]\psi$ with respect to ε, together with the first formula, shows therefore that

$$\mathcal{G}_\mu[\varepsilon\zeta,\beta b]\psi=\mathcal{G}_\mu[0,\beta b]\psi-\varepsilon\mathcal{G}_\mu[0,\beta b](\zeta\mathcal{G}_\mu[0,\beta b]\psi) - \varepsilon\mu\nabla\cdot(\zeta\nabla\psi)+O[\sqrt{\mu}(\varepsilon\sqrt{\mu})^2]. \tag{32}$$

A first order Taylor expansion of $\mathcal{G}_\mu[0,\beta b]\psi$ with respect to b, together with the second formula, gives also

$$\mathcal{G}_\mu[0,\beta b]\psi=\mathcal{G}_\mu[0,0]\psi+\mu\beta\,\mathrm{sech}(\sqrt{\mu}|D|) \times[\nabla\cdot\{b[\mathrm{sech}(\sqrt{\mu}|D|)\nabla\psi]\}] + O[\sqrt{\mu}(\beta\sqrt{\mu})^2]. \tag{33}$$

Let us now define the operators $\mathbf{T}_\mu$ and B_μ as

$$\mathbf{T}_\mu=-\frac{\tanh(\sqrt{\mu}|D|)}{|D|}\nabla \quad \text{and} \quad B_\mu=\mathrm{sech}(\sqrt{\mu}|D|) \times\{b[\mathrm{sech}(\sqrt{\mu}|D|)\cdot]\}. \tag{34}$$

We thus have $\mathcal{G}_\mu[0,0]\psi=\sqrt{\mu}\mathbf{T}_\mu\cdot\nabla\psi$, and Eqs. (32) and (33) show that

$$\mathcal{G}_\mu[\varepsilon\zeta,\beta b]\psi=\sqrt{\mu}\mathbf{T}_\mu\cdot\nabla\psi+\mu\beta\nabla\cdot(B_\mu\nabla\psi) - \varepsilon\mu\mathbf{T}_\mu\cdot\nabla(\zeta\mathbf{T}_\mu\cdot\nabla\psi)-\varepsilon\mu\nabla\cdot(\zeta\nabla\psi) + O[\sqrt{\mu}(\varepsilon\sqrt{\mu})^2,\sqrt{\mu}(\beta\sqrt{\mu})^2]. \tag{35}$$

Remark 4.1: *In the shallow water, weakly nonlinear regime, and with a weakly nonlinear topography [that is,* $\mu\ll 1$, $\varepsilon=O(\mu)$, $\beta=O(\mu)$*], one can deduce from Eq. (35) that*

$$\mathcal{G}_\mu[\varepsilon\zeta,\beta b]\psi=-\mu\nabla\cdot(h\nabla\psi)-\frac{\mu^2}{3}\nabla\cdot\Delta\nabla\psi+O(\mu^3),$$

which can also be obtained using the expansion of the velocity potential used in Sec. III C.

B. Derivation of a fully dispersive model for 2D surface waves over uneven bottoms

We derive here a new system of fully dispersive equations which generalizes to the case of 2D surfaces and nonflat bottoms the systems derived by Matsuno (1D surfaces, flat[28] and uneven[29] bottoms) and Choi[30] and Smith[45] (2D surfaces, flat bottoms).

We first define the horizontal velocity at the surface as $\mathbf{v}_S=(\nabla\Phi)_{|_{z=\varepsilon\zeta}}$, where Φ is the velocity potential given by Eq. (31). By definition of ψ and $\mathcal{G}_\mu[\varepsilon\zeta,\beta b]\psi$, we get

$$\begin{aligned}\nabla\psi&=\mathbf{v}_S+\varepsilon\nabla\zeta(\partial_z\Phi)_{|_{z=\varepsilon\zeta}}\\ &=\mathbf{v}_S+\varepsilon\frac{\mathcal{G}_\mu[\varepsilon\zeta,\beta b]\psi+\varepsilon\mu\nabla\zeta\cdot\nabla\psi}{1+\varepsilon\mu|\nabla\zeta|^2}\nabla\zeta\\ &=\mathbf{v}_S+\varepsilon\sqrt{\mu}(\mathbf{T}_\mu\cdot\mathbf{v}_S)\nabla\zeta+O[(\varepsilon\sqrt{\mu})^2],\end{aligned}$$

where we used Eq. (35) and $\nabla\psi=\mathbf{v}_\mathbf{S}+O(\varepsilon\sqrt{\mu})$ for the last relation. Plugging this relation into Eq. (35), one gets similarly

$$\frac{1}{\sqrt{\mu}}\mathcal{G}_\mu[\varepsilon\zeta,\beta b]\psi=\mathbf{T}_\mu\cdot\mathbf{v}_S+\sqrt{\mu}\beta\nabla\cdot(B_\mu\mathbf{v}_S) - \varepsilon\sqrt{\mu}\mathbf{T}_\mu\cdot(\zeta\nabla\mathbf{T}_\mu\cdot\mathbf{v}_S) - \varepsilon\sqrt{\mu}\nabla\cdot(\zeta\mathbf{v}_S)+O[(\varepsilon\sqrt{\mu})^2,(\beta\sqrt{\mu})^2]. \tag{36}$$

Taking the gradient of the second equation of Eq. (13) and using the above two identities gives therefore the following set of arbitrary depth (or fully dispersive) equations:

$$\partial_t \zeta - \frac{1}{\sqrt{\mu}\nu}\mathbf{T}_\mu \cdot \mathbf{v}_S + \frac{\varepsilon\sqrt{\mu}}{\sqrt{\mu}\nu}[\mathbf{T}_\mu \cdot (\zeta \nabla \mathbf{T}_\mu \cdot \mathbf{v}_s) + \nabla \cdot (\zeta \mathbf{v}_S)] = \frac{\beta\sqrt{\mu}}{\sqrt{\mu}\nu}\nabla \cdot (B_\mu \mathbf{v}_S),$$

$$\partial_t \mathbf{v}_S + \nabla\zeta + \varepsilon\sqrt{\mu}\left(\frac{1}{2\sqrt{\mu}\nu}\nabla|\mathbf{v}_S|^2 - \nabla\zeta \mathbf{T}_\mu \cdot \nabla\zeta\right) = 0, \quad (37)$$

where we recall that $\mathbf{T}_\mu$ and B_μ are defined in Eq. (34) and that $\nu = \tanh(\sqrt{\mu})/\sqrt{\mu}$ [so that one can replace $\sqrt{\mu}\nu$ by 1 in Eq. (37) in deep water].

Remark 4.2: *If we remove the topography term B_μ from these equations, we recover the 2D Eqs. (3.25) and (3.26) derived by Choi.*[30] *If we take the 1D version of Eq. (37) we recover the equations derived by Matsuno [Eqs. (19) and (20) of Ref. 28 for flat bottoms and (4.28) and (4.29) of Ref. 1 for nonflat bottoms].*

Remark 4.3: *Equation (37) is precise up to order $O(\varepsilon\sqrt{\mu}, \varepsilon\sqrt{\beta})$ in deep water (since $\nu \sim \mu^{-1/2}$); one could also use them in shallow water, but they are then precise up to order $O(\varepsilon, \beta)$ only (since $\nu \sim 1$ in shallow water). This is the same accuracy as the one provided by the weakly nonlinear shallow water models (with a weakly nonlinear topography) of Sec. III C; it is therefore not surprising to check that one recovers the Boussinesq system (29) from Eq. (37) by a simple Taylor expansion of the operators $\mathbf{T}_\mu$ and B_μ and by observing that $\bar{\mathbf{v}} = (1 + \mu\frac{1}{3}\Delta)\mathbf{v}_s + O(\mu^2)$.*

In the fully nonlinear regime (with fully nonlinear topography) studied in Sec. III A, Eq. (37) is precise up to order $O(\varepsilon, \beta) = O(1)$ while we saw that the Green–Naghdi equations (26) are precise up to order $O(\mu)$. It is therefore not surprising to check that the shallow water limit of Eq. (37) does NOT give the Green–Naghdi equations (26).

Remark 4.4: *System (37) is "fully dispersive" in the sense that its dispersion relation is the same as for the full water wave equations. This is the case because expansion (36) keeps the nonlocal effects of the Dirichlet–Neumann operator $G_\mu[\varepsilon\zeta, \beta b]\psi$. In Refs. 46–48 the authors make a differential approximation of shallow water type based on Padé approximants; they show that the dispersive properties remain good far beyond the shallow water regime when bathymetric changes are not too strong (in this latter case, the model is more complex and its range of validity much narrower*[49]*).*

V. CONCLUSION

In this paper we have presented a systematic derivation of various 2D asymptotic models for water waves over shallow or arbitrary depth, which allows one to clarify their range of validity. The key point in this systematic derivation is an asymptotic analysis of the Dirichlet–Neumann operator in the different regimes under consideration here. We have also derived a new 2D fully dispersive model, system (37), for small wave steepness which extends to an uneven bottom the approach developed by Matsuno[1] and Choi.[30] We have shown that even though these models remain valid in shallow water, their precision is then far below what can be achieved with the Green–Naghdi or Serre model when fully nonlinear waves are considered ($\varepsilon \sim 1$). Hence, contrary to what it is generally thought[1,28–30] this approach cannot in practice be considered as a unified theory of nonlinear waves because it is not accurate enough for nonlinear shallow water waves ($\epsilon \sim 1$, $\mu \ll 1$). In particular, we have shown that system (37) in the shallow water limit ($\mu \ll 1$) does not correspond to the correct fully nonlinear equations, namely, the Green–Naghdi or Serre equations. Another reason why these fully dispersive water models are not likely to furnish interesting models in shallow water is that there is no obvious shoreline boundary condition for them.

The Green–Naghdi equations represent the appropriate model to describe nonlinear shallow water wave propagation and wave oscillations at the shoreline. For coastal applications, the Green–Naghdi equations can be easily extended to include accurate linear dispersive effects, which allow one to describe shoaling processes in intermediate water depth (see Refs. 6, 23, and 25). Another interest of these equations is that there is a natural shoreline boundary condition given by the flux conservation equation [the flux $h\bar{\mathbf{v}}$ vanishes at the shoreline in the first equation of Eq. (26)]. However, in deeper water the Green–Naghdi model is no more valid and the fully dispersive model (37) must therefore be used.

It follows from these considerations that the asymptotic description of coastal flows requires at least the use of two different models: one for shallow water [e.g., the Green–Naghdi equations (26)] and another one for deeper water [e.g., Eq. (37)]. The numerical coupling of these two models is therefore a natural perspective for further works.

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